

MATH 2040 Lecture 15 (31/10/2016)

§ Dual vector space & existence of T^*

Thm: Given $T: V \rightarrow V$ linear operator on an inner product space (V, \langle, \rangle) , $\dim V < +\infty$.

$\Rightarrow \exists!$ $T^*: V \rightarrow V$ linear called **adjoint of T** st.

\uparrow
unique $\langle Tv, w \rangle = \langle v, T^*w \rangle \quad \forall v, w \in V$

Recall: $[T^*]_{\beta} = [T]_{\beta}^* \quad \beta \text{ O.N.B.}$

Clarify some terminology:

- $T: V \rightarrow W$ linear map / transformation
- $T: V \rightarrow V$ linear operator on V
- $T: V \rightarrow \mathbb{F}$ linear functional on V

$$\mathcal{L}(V, W) := \{ T: V \rightarrow W \text{ linear} \}$$

\uparrow
 $\dim = \dim V \cdot \dim W$

Defⁿ: The **dual vector space of V** is

$$V^* := \{ f: V \rightarrow \mathbb{F} \text{ linear} \} = \mathcal{L}(V, \mathbb{F})$$

Assume: $\dim V < +\infty$ from now on.

Prop: $\dim V^* = \dim V = n \Rightarrow V^* \cong V$ not "canonical" isomorphisms.
 \uparrow
 $+\infty$ \cong \mathbb{F}^n \cong

(Ex: $M_{2 \times 2}(\mathbb{R}) \cong \mathbb{R}^4$ not canonical \leftrightarrow need to choose a basis.)

Motivating example: $V = \mathbb{R}^n$

$$V^* = \{f: \mathbb{R}^n \rightarrow \mathbb{R} \text{ linear}\}$$

$$= \{f(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n\}$$

\uparrow $\dim = n$.

Introduce bases:

$$V = \mathbb{R}^n \quad \beta = \{e_1, \dots, e_n\} \text{ std basis}$$

$$V^* \cong \mathbb{R}^n \quad \beta^* = \{e_1^*, \dots, e_n^*\} \text{ dual basis of } \beta$$

where $e_i^*: \mathbb{R}^n \rightarrow \mathbb{R}$ linear is defined

$$e_i^* \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := x_i \quad \left(\begin{array}{l} i^{\text{th}} \text{ coordinate} \\ \text{functional.} \end{array} \right)$$

Defⁿ: Given a basis $\beta = \{v_1, v_2, \dots, v_n\}$ for V ,

then $\beta^* = \{v_1^*, v_2^*, \dots, v_n^*\}$ is called a dual basis of β for V^*

$$\text{s.t. } \boxed{v_i^*(v_j) = \delta_{ij}} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

i.e. if $v \underset{V}{=} a_1 v_1 + \dots + a_n v_n$, then $v_i^*(v) = a_i$. Ex: Check β^* is a basis.

This gives an isomorphism (once β is chosen)

$$V \begin{array}{c} \xrightarrow{\text{dual}} \\ \cong \\ \xrightarrow{\beta} \end{array} V^*$$

$$V = a_1 v_1 + \dots + a_n v_n \xleftrightarrow{\text{dual}} V^* = a_1 v_1^* + \dots + a_n v_n^*$$

Theorem: $V^{**} \cong V$ "canonically".

Proof: Define a linear map

$$T: V \xrightarrow{\cong} V^{**} = (V^*)^*$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ V & \xrightarrow{\quad} & \hat{V} \end{array} \quad \text{where } \hat{V}(f) := f(v)$$

↑
evaluation at v

↑
 V^*

Check: 1) linearity of T : $\widehat{v_1 + v_2}(f) = f(v_1 + v_2) = f(v_1) + f(v_2) = \hat{v}_1(f) + \hat{v}_2(f) \quad \forall f$

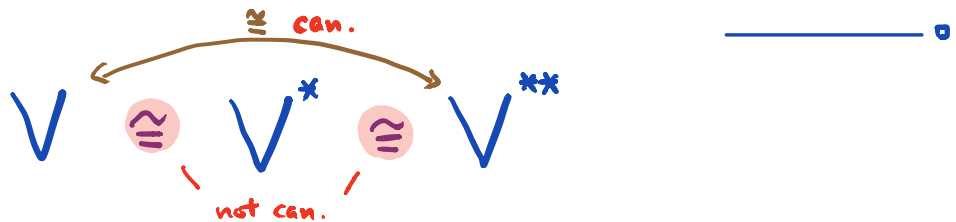
2) T 1-1 ($\Rightarrow T$ onto $\because \dim V = \dim V^{**} = \dim V^*$)

Let $v \in V$ st. $T(v) = 0$

i.e. $0 = \hat{v}(f) := f(v) \quad \forall f \in V^*$

$\Rightarrow v = 0$ (Ex: why?)

So:



Now, look at inner product space

$$(V, \langle \cdot, \cdot \rangle) \quad \dim V < +\infty$$

Thm: $V \cong V^*$ "canonically." (Riesz lemma)

Proof: Want to construct a "natural" isomorphism using the inner product $\langle \cdot, \cdot \rangle$:

$$\begin{array}{ccc} V & \xrightarrow[\text{conj. } \cong]{T} & V^* \\ \downarrow & & \downarrow \\ V & \xrightarrow{\quad} & f_v(\cdot) := \langle \cdot, v \rangle \end{array}$$

Check: 1) $f_v \in V^*$ i.e. $w \mapsto \langle w, v \rangle$ is linear.

(Caution: $w \mapsto \langle v, w \rangle$ is only conjugate linear.)

2) T is conjugate linear.

$$\text{i.e. } T(\alpha v) = \bar{\alpha} T(v)$$

$$f_{\alpha v}(w) := \langle w, \alpha v \rangle$$

$$= \bar{\alpha} \langle w, v \rangle = \bar{\alpha} f_v(w).$$

3) T 1-1 ($\Rightarrow T$ onto)

$$\text{if } f_v(w) = 0 \quad \forall w \in V$$

$$\Rightarrow \langle w, v \rangle = 0 \quad \forall w \in V$$

$$\Rightarrow v = 0 \quad (\text{non-deg. of } \langle \cdot, \cdot \rangle)$$

_____ \square

Cor: Given any $f \in V^*$ on an inner prod. space $(V, \langle \cdot, \cdot \rangle)$,
 then $\exists! v \in V$ s.t. $f(\cdot) = \langle \cdot, v \rangle$.

Now, we can use this to prove the $\exists, !$ of T^* .

$$T : V \rightarrow V \quad (V, \langle \cdot, \cdot \rangle), \dim V < +\infty.$$

define the **adjoint** :

$$T^* : V \xrightarrow{w} V \xrightarrow{w'}$$

Given w , define w' .

$$\text{s.t. } \boxed{\langle Tv, w \rangle = \langle v, T^*w \rangle} \quad \forall v, w \in V. \quad (*)$$

Fix $w \in V$, consider the following

$$f(v) := \langle Tv, w \rangle \quad \forall v \in V$$

- $f \in V^*$ ($\because T$ linear & $\langle \cdot, \cdot \rangle$ linear in 1st slot)
- Riesz Lemma $\Rightarrow f(\cdot) = \langle \cdot, w' \rangle$ for some $w' \in V$

$$\langle Tv, w \rangle =: f(v) = \langle v, w' \rangle = \langle v, T^*w \rangle$$

So, $T^*(w) = w'$ is well-defined!

Check: • T^* is linear \leftarrow (Ex: prove this)

• uniqueness (follows from $(*)$) \leftarrow

_____ \bullet

In general, this happens:

$$T: V \longrightarrow W$$

Question: Can I define T^t ?

2 difficulties: ① $V \neq W$ (even $\dim V \neq \dim W$)

② no \langle, \rangle

Defⁿ: Given a linear transformation $T: V \rightarrow W$.

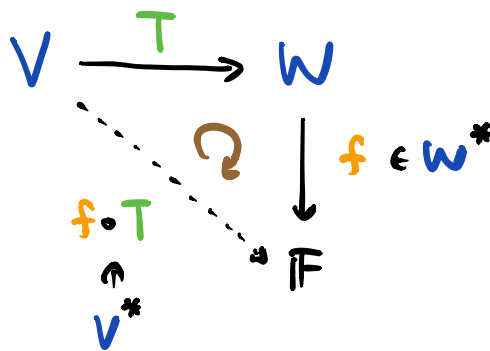
then \exists linear transformation

$$T^t: W^* \longrightarrow V^* \quad \text{"pullback"}$$

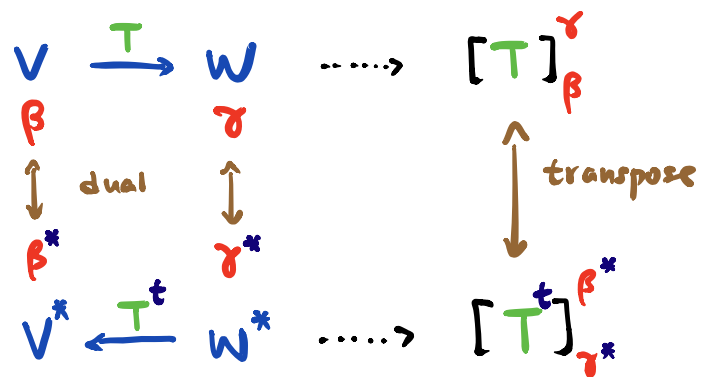
st.

$$\underbrace{T^t(f)}_{V^*}(v) := \underbrace{f}_{W}(T v) \quad \begin{array}{l} \forall f \in W^* \\ \forall v \in V \end{array}$$

Picture:



Prop:



Duality:

