MATH 2040 Lecture 15 (31/10/2016)

## § Dual vector space & existence of $T^*$

**Thm:** Given $T : V \to V$ linear operator on an inner product space $(V, \langle , \rangle)$, $\dim V < +\infty$.

$\Rightarrow \exists! \, T^* : V \to V$ linear called adjoint of $T$ s.t.

unique

$$\langle Tv, w \rangle = \langle v, T^* w \rangle \quad \forall v, w \in V$$

**Recall:** $[T^*]_\beta = [T]_\beta^*$ $\quad \beta$ O.N.B.

Clarify some terminology:

- $T : V \to W$ linear map / transformation
- $T : V \to V$ linear operator on $V$
- $T : V \to \mathbb{F}$ linear functional on $V$

$$\mathcal{L}(V, W) := \{ T : V \to W \text{ linear} \}$$

dim = dim $V$ · dim $W$

**Def$^{\underline{n}}$:** The dual vector space of $V$ is

$$V^* := \{ f : V \to \mathbb{F} \text{ linear} \} = \mathcal{L}(V, \mathbb{F})$$

**Assume:** $\dim V < +\infty$ from now on.

**Prop:** $\dim V^* = \dim V = n \Rightarrow$ $V^* \cong V$, $V^* \cong \mathbb{F}^n \cong V$ not "canonical" isomorphisms.

$\underset{+\infty}{\wedge}$

( **Eg.** $M_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$ not canonical $\Leftrightarrow$ need to choose a basis. )

Motivating example: $V = \mathbb{R}^n$

$$V^* = \{ f : \mathbb{R}^n \to \mathbb{R} \text{ linear} \}$$

$$= \{ f(x_1, \dots, x_n) = a_1 x_1 + \cdots + a_n x_n \}$$

$\dim = n$.

Introduce bases:

$V = \mathbb{R}^n$ $\quad \beta = \{ e_1, \dots, e_n \}$ std basis

$V^* \cong \mathbb{R}^n$ $\quad \beta^* = \{ e_1^*, \dots, e_n^* \}$ dual basis of $\beta$

where $e_i^* : \mathbb{R}^n \to \mathbb{R}$ linear is defined

$$e_i^* \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := x_i \quad (i^{th} \text{ coordinate functional.})$$

**Def$^{\underline{n}}$:** Given a basis $\beta = \{ v_1, v_2, \dots, v_n \}$ for $V$.

then $\beta^* = \{ v_1^*, v_2^*, \dots, v_n^* \}$ is called a dual basis of $\beta$ for $V^*$

s.t. $$\boxed{v_i^*(v_j) = \delta_{ij}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

i.e. if $\underset{\overset{\wedge}{V}}{v} = a_1 v_1 + \cdots + a_n v_n$, then $v_i^*(v) = a_i$.

**Ex:** Check $\beta^*$ is a basis.

This gives an isomorphism (once $\beta$ is chosen)

$$\begin{array}{ccc} V & \xleftrightarrow[\cong]{\text{dual}} & V^* \\ \beta & & \beta^* \end{array}$$

$$v = a_1 v_1 + \cdots + a_n v_n \xleftrightarrow{\text{dual}} v^* = a_1 v_1^* + \cdots + a_n v_n^*$$

**Theorem:** $V^{**} \cong V$ "canonically".

**Proof:** Define a linear map

$$T : V \xrightarrow{\cong} V^{**} = (V^*)^*$$
$$\quad\; v \longmapsto \hat{v} \quad \text{where} \quad \hat{v}(f) := f(v), \quad f \in V^*$$

$\hat{v}$: evaluation at $v$

Check: 1) linearity of $T$: $\widehat{v_1+v_2}(f) = f(v_1+v_2) = f(v_1)+f(v_2)$
$= \hat{v}_1(f) + \hat{v}_2(f) \quad \forall f$

2) $T$ 1-1 ($\Rightarrow T$ onto $\because \dim V = \dim V^{**} = \dim V^*$)

Let $v \in V$ s.t. $T(v) = 0$

i.e. $0 = \hat{v}(f) := f(v) \quad \forall f \in V^*$

$\Rightarrow v = 0$. (Ex: why?)

$\square$

So:

$$V \overset{\text{not can.}}{\cong} V^* \overset{\text{not can.}}{\cong} V^{**}, \qquad V \overset{\text{can.}}{\cong} V^{**}$$

Now, look at inner product space

$$( V , \langle , \rangle ) \qquad \dim V < +\infty$$

Thm: $V \overline{\cong} V^*$ "canonically." (Riesz lemma)

Proof: Want to construct a "natural" isomorphism using the inner product $\langle , \rangle$:

$$\begin{array}{ccc} V & \xrightarrow[\text{conj.} \cong]{T} & V^* \\ \Psi & & \Psi \\ v & \longmapsto & f_v(\cdot) := \langle \cdot , v \rangle \end{array}$$

Check: 1) $f_v \in V^*$ i.e. $w \overset{f_v}{\longmapsto} \langle w, v \rangle$ is linear.

(Caution: $w \longmapsto \langle v, w \rangle$ is only conjugate linear.)

2) $T$ is conjugate linear.

i.e. $T(\alpha v) = \bar{\alpha} T(v)$

$$f_{\alpha v}(w) := \langle w, \alpha v \rangle$$
$$= \bar{\alpha} \langle w, v \rangle = \bar{\alpha} f_v(w).$$

3) $T$ 1-1 ($\Rightarrow$ $T$ onto)

if $f_v(w) = 0 \quad \forall w \in V$

$\Rightarrow \langle w, v \rangle = 0 \quad \forall w \in V$

$\Rightarrow v = 0$ (non-deg. of $\langle , \rangle$)

$\square$

Cor: Given any $f \in V^*$ on an inner prod. space $(V, \langle , \rangle)$, then $\exists!\ v \in V$ s.t. $f(\cdot) = \langle \cdot , v \rangle$.

Now, we can use this to prove the $\exists, !$ of $T^*$.

$$T : V \longrightarrow V \qquad (V, \langle , \rangle),\ \dim V < +\infty.$$

define the adjoint :

$$T^* : \overset{w}{\underset{\uparrow}{V}} \longrightarrow \overset{w'}{\underset{\uparrow}{V}}$$

Given w, define w'.

s.t. $\boxed{\langle Tv, w \rangle = \langle v, T^* w \rangle}$ (*) $\forall\ v, w \in V$.

Fix $w \in V$, consider the following

$$f(v) := \langle Tv, w \rangle \qquad \forall\ v \in V$$

- $f \in V^*$ ($\because$ $T$ linear & $\langle , \rangle$ linear in 1st slot)
- Riesz Lemma $\Rightarrow$ $f(\cdot) = \langle \cdot , w' \rangle$ for some $w' \in V$

$$\langle Tv, w \rangle =: f(v) = \langle v, w' \rangle = \langle v, T^* w \rangle$$

So, $T^*(w) = w'$ is well-defined!

Check: · $T^*$ is linear ← (Ex: prove this)

· uniqueness (follows from (*))

In general, this happens:

$$T : V \longrightarrow W$$

Question: Can I define $T^t$?

2 difficulties: ① $V \neq W$ (even $\dim V \neq \dim W$)

② no $\langle , \rangle$

Def$^{\underline{n}}$: Given a linear transformation $T : V \to W$.

then $\exists$ linear transformation

$$T^t : W^* \longrightarrow V^* \qquad \text{"pullback"}$$

s.t. $$\underbrace{T^t(f)}_{V^*}(v) := f(\underset{\substack{\uparrow \\ W}}{Tv}) \qquad \forall f \in W^*, \ \forall v \in V$$

Picture:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[dr, dashed, "f \circ T"'] & W \arrow[d, "f \in W^*"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathbb{F}
\end{tikzcd}$$

$$\underset{\substack{\uparrow \\ V^*}}{f \circ T}$$

**Prop:**

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[r, dotted] & {[T]_\beta^\gamma} \arrow[d, leftrightarrow, "\text{transpose}"] \\ V^* & W^* \arrow[l, "T^t"'] \arrow[r, dotted] & {[T^t]_{\gamma^*}^{\beta^*}} \end{tikzcd}$$

$$\beta \overset{\text{dual}}{\longleftrightarrow} \beta^* \qquad \gamma \longleftrightarrow \gamma^*$$

**Duality:**

$$V = \mathbb{R}^n \xleftrightarrow{\text{dual}} \mathbb{R}^n = V^*$$

$$\rotatebox{90}{$\in$}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \xleftrightarrow{\text{dual}} (x_1 \cdots\cdots x_n)$$

columns $\xleftrightarrow{\text{dual}}$ rows